Let P be a principle G-bundle over an oriented manifold X:

$$G \circlearrowright P \longrightarrow X$$

By convention, G has a right action on P.

Let $\mathcal{A}(P)$ denote the space of connections on P.

Let $\mathcal{A}_{flat}(P)$ denote the space of flat connections on P.

Let $\mathcal{G}(P)$ denote the gauge group. This is defined as the space of maps

$$u : P \longrightarrow G$$

that are equivariant:

$$u(p \cdot g) = g^{-1} u(p) g$$

for all $g \in G$ and $p \in P$. Then, $u \in \mathcal{G}(P)$.

Then $\mathcal{G}(P)$ acts on $\mathcal{A}(P)$ and on $\mathcal{A}_{flat}(P)$.

The action of $u \in \mathcal{G}(P)$ on $A \in \mathcal{A}(P)$ is denoted by $u^* A$.

Fact/theorem:

flat connections modulo gauge transformations — group homomorphisms modulo conjugation

$$\bigsqcup_{[P]} \mathcal{A}_{flat}(P) / \mathcal{G}(P) \cong \hom(\pi_1(X), G)/G,$$

where the action of G on the r.h.s. is coming from conjugation (by the inverse) and the disjoint union is over the set of equivalence classes of bundles $P \to X$.