Preliminary notes on TQFTs
(Source: R. Dijkgraaf, 9703136)

Simplest type of 2d QFT = TQFT w/ the property that all amplitudes are independent of the local Riemannian structure.

AXIOMS OF TOPOLOGICAL FIELD THEORY

Category: consists of a set of objects x and a set of arrows or morphisms $f: x \to y$ that satisfy an associative composition law; i.e. given two arrows $f: x \to y$ and $g: y \to z$, we can form the composite arrow $g \circ f: x \to z$ such that

$$(h \circ g) \circ f = h \circ (g \circ f)$$

A category further presumes that for each object x, an arrow $1_x : x \to x$ exists, satisfying

$$f \circ 1_x = 1_y \circ f = f.$$

Functor: A functor between two categories is a map that maps objects to objects, morphisms to morphisms, that respects all relations.

example: the category Vect of complex, possibly graded vector spaces, where

- objects are obviously vector spaces
- morphisms correspond to linear maps between the vector spaces

Vect is an example of an ABELIAN TENSOR CATEGORY, where the objects and morphisms can also be multiplied using the associative and commutative tensor product $\otimes$, and have an inverse, the linear dual V^*.

If we have maps

$$\Phi_1 : V_1 \to W_1$$

and

$$\Phi_2 : V_2 \to W_2$$

we can form the tensor product map

$$\Phi_1 \otimes \Phi_2 : V_1 \times V_2 \longrightarrow W_1 \times W_2$$

The unit is $\mathbb{C}$.

- **Man(d)**: category of d-dimensional manifolds, where the objects are smooth, compact, oriented manifolds X (not defined up to isomorphism, and thus parametrization in local coordinates) equipped with a given and where the morphisms

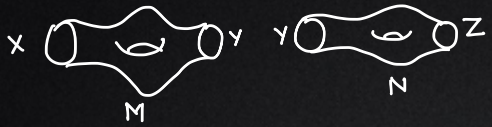
$$M : X \longrightarrow Y$$

are BORDISMS. That is, M is a smooth, oriented mfd. of dim. $(d+1)$ s.t. it has two bdary components X and Y satisfying

$$\partial M = (-X) \sqcup Y$$

Composition law of bordisms

$M : X \to Y$, $N : Y \to Z$ are bordisms



Then,

$N \circ M : X \to Z$ is a bordism

Obtained by identifying the two bdary components Y and $-Y$. The identification can be done in a unique way because both copies of Y come with a parametrization.

Identity Morphism: the identity morphism is given by the cylinder

$$1_X : X \times [0,1]$$

THESE DEFINITIONS MAKE **MAN (d)** INTO A CATEGORY.

Why is the identity morphism given by $1_X : X \times [0,1]$?

Note that with this definition, the "incoming" state is X and the "outgoing" state is also X and nothing happens "in between" $\longrightarrow$ this is the expected action of the identity morphism.

① $X \times [0,1]$ has dimension $(d+1)$

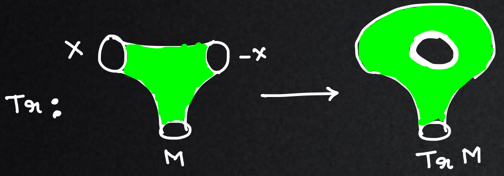
② $X \times [0,1]$ has 2 boundary components, i.e. the copies of X at 0 and 1, the end-points of the closed interval $[0,1]$

③ It remains to be shown that $\partial(X \times [0,1])$ is the disjoint union of $-X$ and X (where $-X$ denotes the mfd. X with opposite orientation). But this follows from the following figure that shows the opposite directions of the natural normals at the two points.

(As of Oct. 22, 2018, I do not know of a better proof.)

There is one extra operation that we include that is not standard in categories. This is the partial trace : we want to be able to glue two boundary components of a single irreducible manifold : if the two boundary components of M contain a common factor X we want to define the partial trace :

$$\mathrm{Tr}_X : M \longrightarrow \mathrm{Tr}_X M$$



Man(d) is also a tensor category with the product given by the disjoint union $\sqcup$ and the empty set $\emptyset$.

Topological Field Theory : A $(d+1)$-dimensional topological field theory can now be defined as a functor

$$\Phi : \mathbf{Man(d)} \longrightarrow \mathbf{Vect}$$

from the category of d-manifolds to the category of vector spaces, satisfying certain extra properties. To any space X, we associate a vector space V_X,

$$\Phi : X \longrightarrow V_X$$

and to any bordism M a linear map Φ_M

$$(M : X \longrightarrow Y) \Rightarrow (\Phi_M : V_X \longrightarrow V_Y).$$

The fact that Φ is a functor tells us that the amplitudes Φ_M should satisfy a factorization law

$$\Phi_{N \circ M} = \Phi_N \circ \Phi_M$$

- If the relevant operator is trace-class (or when all vector spaces V_X are finite-dimensional) we have an additional condition

$$\Phi_{\mathrm{Tr}_X M} = \mathrm{Tr}_{V_X} \Phi_M .$$

- Since both categories **Man(d)** and **Vect** are abelian tensor categories, we further demand that Φ respects the products $\otimes$ and $\sqcup$

$$V_{X \sqcup Y} = V_X \otimes V_Y , \quad V_{-X} = V_X^* , \quad V_\emptyset = \mathbb{C}$$

and :

$$\Phi_{M \sqcup N} = \Phi_M \otimes \Phi_N , \quad \Phi_{-M} = \Phi_M^* , \quad \Phi_\emptyset = \mathbb{C}$$

Path-integral motivation for the existence of Φ_M amplitudes

$\phi(x)$: local set of fields in the theory

V_X : Hilbert space

$\Psi(\phi_X)$: state in V_X, a wavefunction on the space of field configurations $\phi_X(x)$ on the spacelike manifold X.

Path integral on M with fixed values ϕ_X, ϕ_Y at the boundaries X and Y then gives the kernel $K_M(\phi_Y, \phi_X)$ of the evolution operator Φ_M

$$K_M(\phi_Y, \phi_X) = \int_{\substack{\phi|_X = \phi_X \\ \phi|_Y = \phi_Y}} \mathcal{D}\phi \, e^{-S(\phi)}$$

Transition amplitude then relates an "incoming" wavefxn $\Psi_{in}(\phi_X)$ to an "outgoing" wavefxn $\Psi_{out}(\phi_Y)$ as

$$\Psi_{out}(\phi_Y) = \int \mathcal{D}\phi_X \, K_M(\phi_Y, \phi_X) \, \Psi_{in}(\phi_X)$$

Gluing law corresponds to the composition law in QM

$$K_{N\circ M}(\phi_Z, \phi_X) = \int \mathcal{D}\phi_Y \, K_N(\phi_Z, \phi_Y) K_M(\phi_Y, \phi_X)$$

evolving along M and then evolving along N is equivalent to evolving along $N \circ M$.